

Section 11.2: The Shapley-Shubik Model

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We want to measure how “powerful” certain voters are in a weighted voting system. The first way we will attempt to measure power is using the *Shapley-Shubik* model.

Definition. A **voting permutation** is an ordered list of all the voters in a voting system.

Assumptions of the Shapley-Shubik model.

- (1) For every issue to be voted on there is a *spectrum of opinion* where voters can be ordered in some way. The ordering of the voters corresponds to a voting permutation.
- (2) Every issue can be placed on the spectrum where all voters at or above the placement will vote “Yes” and all voters below the placement will vote “No.”
- (3) Given any issue, every voting permutation has the same chance of reflecting the spectrum of opinion on that issue.

Exercise 1. In the fictitious state of Kerplexistan, there are five counties in its Senate with voting weights that reflect each counties population. The counties are Andorra (Voter 1), Backus (Voter 2), Capri (Voter 3), Dribble (Voter 4), and Eon (Voter 5) with the respective voting weights 5, 3, 2, 2, 5. The quota is 9. That is, the weighted voting system is $[9; 5, 3, 2, 2, 5]$.

- (a) In the Kerplexistan Senate, is there any county with veto power? Is there any county that is a dummy voter?
- (b) The Senate is voting on a new income tax rate. Each county has a maximum rate that they are willing to vote for. These rates are summarized in the table below:

Andorra	Backus	Capri	Dribble	Eon
1%	5%	0%	4%	7%

Write the corresponding voting permutation on the tax issue where the county willing to have the largest maximum tax rate is on the left.

- (c) Would a 1% tax rate be passed by the Senate? Would a 2% tax rate be passed by the Senate?
- (d) What is the largest tax rate that could be passed by the Senate?
- (e) Consider the following definition:

Definition. The first voter in a voting permutation who, when joined by those coming before her or him, would have enough voting weight to win, is the **pivotal voter** in the permutation.

Which county is the pivotal voter in the voting permutation you found in this tax example?

- (f) Suppose the maximum rates each county would vote for was

Andorra	Backus	Capri	Dribble	Eon
1%	5%	3%	8%	0%

Which county is now the pivotal voter?

- (g) Find a table of maximum rates so that Backus is the pivotal voter.

Exercise 2. Consider a weighted voting system $[8; 6, 4, 2]$.

- (a) There are six different voting permutations for this voting system. Find them labeling the voters as A , B , and C (A has 6 votes, B has 4 votes, C has 2 votes).
- (b) For each of the voting permutations for this voting system, find the pivotal voter.
- (c) How many times is Voter A the pivotal voter? How many times is Voter B the pivotal voter? How many times is Voter C the pivotal voter?
- (d) Consider the following definition:

Definition. For a weighted voting system, the **Shapley-Shubik power index** of a voter is computed by taking the number of permutations in which that voter is the pivotal voter and dividing by the number of all voting permutations.

Compute the Shapley-Shubik power index for each voter of the voting system $[8; 6, 4, 2]$.

- (e) You should have found that Voters B and C have the same Shapley-Shubik power index, despite having a different numbers of votes. Does this make sense, or does this expose a flaw in the definition of the power index? Explain your answer.
- (f) According to the Shapley-Shubik power index, how many times more powerful is Voter A than either Voter B or C ?